MATH 5061 Problem Set 4[1]
Due date: Mar 25, 2024

Problems: (Please hand in your assignments by submitting your PDF via email. **Late submissions will not be accepted.**)

Throughout this assignment, we use (M, g) to denote a smooth n-dimensional Riemannian manifold with its Levi-Civita connection ∇ unless otherwise stated. The Riemann curvature tensor (as a $(0, 4)$-tensor) of (M, g) is denoted by R.

1. Let (M, g) be a Riemannian manifold. Fix $p \in M$.
   (a) Suppose the exponential map $\exp_p$ is defined on the whole tangent space T_pM. Prove that for any $q \in M$, there exists a geodesic γ joining p to q such that $L(\gamma)$ realized the Riemannian distance $\rho(p, q)$ between p and q. Use this to show that (M, ρ) is complete as a metric space.
   (b) Prove the converse of (a), i.e. suppose (M, ρ) is a complete metric space, show that $\exp_p$ is well-defined on T_pM.
2. Prove that every Jacobi field V along a geodesic γ in (M, g) arises from the variation vector field of a 1-parameter family of geodesics.
3. A vector field $X \in \Gamma(TM)$ is said to be a *Killing vector field* if $\mathcal{L}_X g = 0$.
   (a) Suppose M is compact. Show that X is a Killing vector field if and only if the flow $\{\varphi_t\}$ of diffeomorphisms of M generated by X consists of isometries of (M, g).
   (b) Prove that any Killing vector field X restricts to a Jacobi field on every geodesic in M.
   (c) Suppose M is connected. Show that a Killing vector field X on M which vanishes at some $p \in M$ and $\nabla_Y X(p) = 0$ for all $Y(p) \in T_pM$ must vanish everywhere on M.
4. Show that Synge theorem does not hold in odd dimensions.

[1] Last revised on March 10, 2024